Large-amplitude electromagnetic waves in relativistic plasmas

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The propagation of linearly polarized large-amplitude electromagnetic waves in relativistic plasmas is studied in the framework of the Akhiezer-Polovin model. Different forms of the basic equations are reviewed and important solutions are presented for small and critical plasma densities. The well-known periodic solutions are generalized to quasiperiodic solutions taking account of additional electrostatic oscillations.

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INTRODUCTION

In the last decade the interaction between laser pulses of relativistic intensities and matter has become a field of growing interest. The introduction of modern laser systems with intensities above the relativistic intensity threshold of roughly $10^{18} W/cm^2$ opened the door to a regime in which electrons at the laser focus oscillate at relativistic velocities [1]. Generally, the first part of an ultra-intense pulse ionizes a material, so that the propagation of the rest of the pulse takes place in a plasma. The relativistic nonlinearities in the laser-plasma interaction and the violation of the plasma local charge neutrality lead to a whole bunch of new phenomena, like self-focussing and self-modulation [2, 3], filamentation [4, 5], higher harmonics generation [6] and particle acceleration [7–10] as well as pulse compression and attosecond pulse generation [11–14].

In this work, the problem of plane linearly polarized ultra-intense electromagnetic waves in cold plasmas is studied in the framework of the Akhiezer-Polovin model. This model is completely relativistic and allows one to study the propagation properties of waves from small up to ultra-large intensities in detail. Although it is a plane wave model it can also approximately be used for the description of sufficiently long pulses. Of special interest is the nonlinear coupling between transverse electromagnetic and longitudinal electrostatic modes. This coupling yields either periodic or quasiperiodic waves.

Periodic waves result if the the longitudinal component oscillates with a multiple of the laser frequency ω. A well-known periodic solution is the figure-eight trajectory of a single relativistic electron in an electromagnetic wave. Most previous work on the Akhiezer-Polovin model focused on periodic waves [15–24]. However, periodic waves exist only for properly adjusted initial conditions. For general initial conditions an additional electrostatic oscillations is excited. Physically, such an oscillation is induced by the relativistic light pressure (or the Lorentz-force) of the laserbeam at the plasma interface. The characteristic frequency of the electrostatic oscillation is the plasma frequency $\omega_{\rm p}$. The result of the nonlinear coupling of the electromagnetic wave to this electrostatic oscillation is a quasiperiodic wave. Because the two frequencies ω and $\omega_{\rm p}$ are generally incommensurable the wave is no longer periodic and the electron orbits are in general non-closed [25, 26]. In this work, the Akhiezer-Polovin model of electromagnetic wave propagation in relativistic plasmas will briefly be reviewed and recent extensions to the quasiperiodic regime will be discussed.

Throughout this work dimensionless quantities are used. Time, coordinates and electromagnetic potentials are normalized with $1/\omega$, c/ω and $m_e c^2/e$, respectively. In addition, the electron density n is normalized with the density n_0 of the ions at rest.

BASIC EQUATIONS

The basic equations for the investigation of the propagation of plane electromagnetic waves in cold plasmas have been introduced by Akhiezer and Polovin [15]. In this model the electrons are treated relativistically, the ions are assumed stationary, and thermal effects are neglected. The model equations follow from Maxwell equations and the relativistic equations of motion for a cold electron fluid. Under the assumption of plane waves with frequency ω, wave number k and a constant phase velocity $v_{ph} = \omega/k$ propagating in the x-direction all quantities depend only on the phase $\hat{\phi} = t - x/v_{ph}$. In the case of linear polarization in the y-direction the equations reduce to two nonlinear, coupled differential equations. In the basic set of equations Akhiezer and Polovin use the longitudinal and transverse electron momenta p_x and p_y as independent variables [15, 16],

$$p_{\rm y}'' + \frac{\alpha v_{\rm ph}}{v_{\rm ph}\gamma - p_{\rm x}} p_{\rm y} \;=\; 0, \qquad (1a)$$

$$(v_{\rm ph} p_{\rm x} - \gamma)'' + \frac{\alpha (v_{\rm ph}^2 - 1)}{v_{\rm ph}\gamma - p_{\rm x}} p_{\rm x} \;=\; 0, \qquad (1b)$$

where $\alpha = \omega_p^2 v_{ph}^2/(v_{ph}^2 - 1)$ is a constant, $\gamma = (1 + p_x^2 + p_y^2)^{1/2}$ is the relativistic Lorentz factor, $\boldsymbol{p} = \gamma \boldsymbol{v}$ is the dimensionless relativistic momentum, and primes denote derivatives with respect to the phase. These equations describe the full relativistic coupling between the transverse electromagnetic and the longitudinal electrostatic oscillations. The equations are valid for arbitrary laser intensities ranging from the non-relativistic up to the ultra-relativistic regime. With this approach Akhiezer and Polovin [15] investigated relativistic longitudinal oscillations and circularly polarized electromagnetic waves. The latter obey the famous relativistic dispersion relation

$$c^2k^2 = \omega^2 - \frac{\omega_p^2}{\gamma}.$$

Furthermore, they briefly studied linearly polarized waves, considering the lowest order vacuum solution as well as a critical density solution. In the weakly relativistic regime Lünow [17] derived in addition a solution up to the third-order in the amplitude. On the other hand Kaw and Dawson [16] and Max and Perkins [18] focused their attention to the almost-transverse solution at critical densities and derived a weakly relativistic as well as an ultra-relativistic solution. In later work ion motion was also included [27].

Another form of Eqs. (1) can be derived using the electromagnetic potentials A_y and φ instead of the momenta [26]:

$$A_y'' + \frac{\alpha v_{ph}}{\sqrt{\varphi^2 + (1 + A_y^2)(v_{ph}^2 - 1)}} A_y = 0, \tag{2a}$$

$$\varphi'' + \frac{\alpha v_{ph}}{\sqrt{\varphi^2 + (1 + A_y^2)(v_{ph}^2 - 1)}} \varphi = \alpha. \tag{2b}$$

The potentials can be related to the momenta by $A_y = p_y$ and $\varphi = \gamma - v_{ph} p_x$. Similar equations were used by Sprangle *et al.* [21, 22, 28, 29] and later by Zhmoginov and Fraiman [23] to investigate the generation of higher harmonics in the transverse component for the case of small plasma densities. In later work Borovsky *et al.* also considered non-periodic solutions [26, 30, 31].

Instead of the momenta or the potentials as a function of the phase $\hat{\phi}$ as in (1) or (2) it can also be convenient to use Lagrangian coordinates as a function of the proper time τ,

$$\eta(\tau) = y - y_0, \tag{3a}$$

$$\xi(\tau) = x - x_0. \tag{3b}$$

For these variables the equations of Akhiezer and Polovin can be transformed into a simpler set of equations [24]:

$$\ddot{\eta} + \alpha\gamma\eta = \alpha/v_{ph}\eta p_x, \tag{4a}$$

$$\ddot{\xi} + \omega_p^2\gamma\xi = -\alpha/v_{ph}\eta p_y, \tag{4b}$$

where the dots denote derivatives with respect to the proper time, and $p_y = \dot{\eta}$ and $p_x = \dot{\xi}$ are the momenta.

This set of differential equations corresponds to a system of coupled relativistic harmonic oscillators and the fundamental symmetry between η and ξ is clarified. In this form the solutions can easily be expanded and higher order dispersion relations can be calculated for small and critical plasma densities [24]. Similar equations were derived by Lünow [17] but were used only to investigate the one-dimensional plasma oscillation ($\eta = 0$).

Another approach was first introduced by Winkles *et al.* [32] and was later used extensively by Clemmow [33]. They considered a system $\tilde{S}$ moving with the velocity $-1/v_{ph}$ relative to the laboratory system. In $\tilde{S}$ the ions are streaming with $1/v_{ph}$. All quantities are then space-independent and the magnetic field as well as the density modulation disappears. The only nonlinearity left is the relativistic factor γ [33]:

$$\frac{d^2}{d\tilde{t}^2}\tilde{p}_y + \frac{\omega_p^2}{\tilde{\gamma}}\tilde{p}_y = 0, \tag{5a}$$

$$\frac{d^2}{d\tilde{t}^2}\tilde{p}_x + \frac{\omega_p^2}{\tilde{\gamma}}\tilde{p}_x = -\frac{\omega_p^2}{v_{ph}}, \tag{5b}$$

All tilded expressions are connected with the laboratory frame expressions by a Lorentz transformation. In many cases the solution is simplified by this approach [19, 33]. Many results were summarized by Decoster [20]. Nevertheless, this system is not convenient in the case of a vanishing plasma density since that implies $v_{ph} \to 1$.

It should be noted, that all the different basic equations presented above are equivalent. They are all nonlinear and have to be solved self-consistently. Which set of basic equations is most convenient depends on the specific problem. The solution for a set of basic variables can be tranformed to other basic variables and to other reference frames. From a solution all other quantities like the electron density n and the fields can be calculated. In this work, all results are presented in terms of the potenials A_y and φ. Futhermore the phase $\hat{\phi}$ is normalized by $1/\sqrt{\alpha}$, $\phi = \sqrt{\alpha}\hat{\phi}$. The electron density, the particle energy (Lorentz factor) and the longitudinal momentum can then be calculated by

$$n - 1 = \frac{1}{v_{ph}^2 - 1}\left(1 - \frac{v_{ph}\varphi}{\sqrt{\varphi^2 + (v_{ph}^2 - 1)(1 + A_y^2)}}\right),$$

$$\gamma = \frac{1}{v_{ph}^2 - 1}\left(v_{ph}\sqrt{\varphi^2 + (v_{ph}^2 - 1)(1 + A_y^2)} - \varphi\right),$$

$$p_x = \frac{1}{v_{ph}^2 - 1}\left(\sqrt{\varphi^2 + (v_{ph}^2 - 1)(1 + A_y^2)} - v_{ph}\varphi\right).$$

In the following, the solutions of the Akhiezer and Polovin problem are analytically and numerically investigated for small and critical plasma densities. For small plasma densities the phase velocity v_{ph} is just a little greater than one. In this case the plasma frequency ω_p [24] or the small parameter,

$$\epsilon = \sqrt{v_{ph}^2 - 1}, \tag{7}$$

is an appropriate expansion parameter [26]. At the critical density the phase velocity v_{ph} approaches infinity, but the wave number k of an electromagnetic wave approaches zero, $k \to 0$. The latter applies also to the parameter

$$\kappa = \sqrt{\frac{1}{v_{ph}^2 - 1}}. \tag{8}$$

This parameter is just the inverse of the small parameter ϵ. For critical densities it is sometimes convenient to use $\kappa\varphi$ instead of φ as a variable because $\kappa\varphi = -p_x + \mathcal{O}(\kappa^2)$. In the subsequent analysis without loss of generality the initial condition $A_y = 0$ at $\phi = 0$ is used.

PERIODIC WAVES

For periodic solutions the plasma electron trajectories resemble at small plasma densities the famous closed eight-like vacuum trajectory (Fig. 1). The plasma density induces just a small correction to the vacuum solution because for periodic solutions no electrostatic oscillation is excited. An expansion up to order ϵ^4 yields the result (eight-like solution)

$$\begin{aligned} A_y &= -a_0 \sin(f) \qquad \text{(9a)}\\ &+\epsilon^2 \left(-\frac{3}{256} a_0^3 \varphi_0^{-2} \sin(3f) - \frac{1}{16} a_0^3 \varphi_0^{-2} \sin(f) \right)\\ &+\epsilon^4 \Bigg(-\frac{85}{2^{17}} a_0^5 \varphi_0^{-4} \sin(5f)\\ &+\frac{3(149a_0^2 + 384)}{2^{17}} a_0^3 \varphi_0^{-4} \sin(3f) - a_4 \sin(f) \Bigg),\\ \varphi &= \varphi_0 + \epsilon^2 \left(\frac{1}{16} a_0^2 \varphi_0^{-1} \cos(2f) \right) \qquad \text{(9b)}\\ &+\epsilon^4 \Bigg(\frac{19}{2^{13}} a_0^4 \varphi_0^{-3} \cos(4f)\\ &-\frac{23a_0^2 + 96}{2^{11}} a_0^2 \varphi_0^{-3} \cos(2f) \Bigg), \end{aligned}$$

where $f = \phi/\sqrt{\langle\gamma\rangle}$, $\varphi_0 = \langle\gamma\rangle = \sqrt{1 + \frac{1}{2}a_0^2}$ and a_4 is a higher order correction to the $\sin(f)$-component. This expansion generalizes our previous result [24] by one further order in ϵ^2. The corrections comprise the generation of higher harmonics [22, 29] but also corrections to the fundamental oscillation. The expansion parameter ϵ is proportional to $\omega_p/\sqrt{\langle\gamma\rangle}$, which is the effective relativistic plasma frequency. A dispersion relation for this solution can be calculated from the constraint that the solution is 2π-periodic in the phase $\hat{\phi}$ [24].

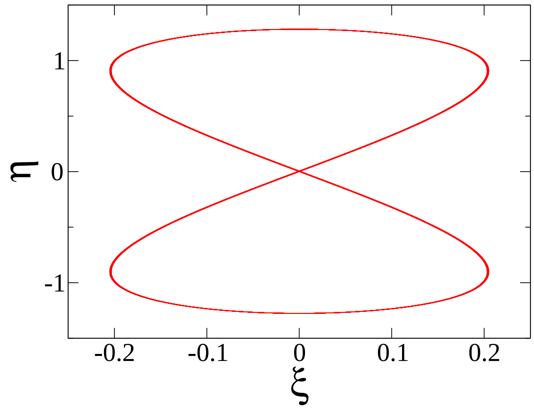

FIG. 1: Periodic eight-like trajectory for small plasma densities. ($\epsilon = 0.1$, $\varphi(0) = 2.34$, $A_y'(0) = -1.96$, *i.e.* $a_0 = 3$).

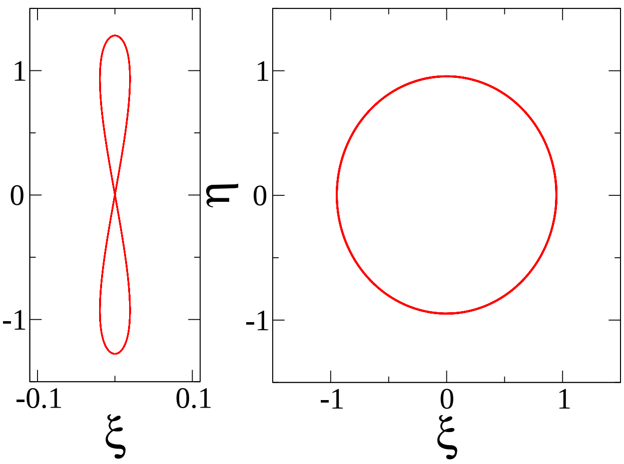

FIG. 2: Periodic solutions near the critical density ($\kappa = 1/\epsilon = 0.05$ and $a_0 = 3$), left: almost-transverse eight-like trajectory ($\kappa\varphi(0) = 0.114$, $A_y'(0) = -1.96$), right: circle-like trajectory ($\kappa\varphi(0) = -2.78$, $A_y'(0) = -1.73$).

For near critical plasma densities, on the other hand, there exist two important classes of periodic solutions. The first class comprises the almost-transverse eight-like solutions [15, 16, 18, 19]. The second class consists of circle-like solutions which are however still linearly polarized in the plane perpendicular to the propagation direction [15, 24] (Fig. 2).

The almost-transverse eight-like solution is a direct generalization of the small density solution (9). Numerical investigations indicate that an eight-like solution exists for every plasma density from small up to critical densities. However, for critical plasma densities analytic solutions are only known for the ultra-relativistic limit [18] and for the weakly relativistic limit [16]. In terms of the potentials the weakly relativistic solution can in the lowest order be summed up as

$$A_y = -a_0 \sin(\phi), \tag{10a}$$
$$\kappa\varphi = \kappa(1 + \frac{a_0^2}{4} + \frac{1}{12}a_0^2 \cos(2\phi)). \tag{10b}$$

The resulting trajectory is very narrow in comparison to the small density case.

The circle-like solutions of the second class result from a different coupling of the longitudinal and the transverse mode. For this solutions both A_y and φ oscillate at the fundamental frequency. A calculation up to the order κ yields the result [24]

$$A_y = \mp a_0 \sin(f) \mp \kappa \frac{a_0^2 \langle\gamma\rangle}{2(2a_0^2+3)} \sin(2f), \tag{11a}$$
$$\kappa\varphi = -a_0 \cos(f) - \kappa \frac{a_0^2 \langle\gamma\rangle}{2(2a_0^2+3)} \cos(2f) + \kappa \frac{\langle\gamma\rangle (7a_0^2+6)}{2(2a_0^2+3)}, \tag{11b}$$

where again $f = \phi/\sqrt{\langle\gamma\rangle}$, but this time $\langle\gamma\rangle = \sqrt{1+a_0^2}$. At a first glance, it seems that the circular solution has nothing in common with the eight-like solution. However, in [24] it has been shown that for intermediate plasma densities there is a transition point from the circular solution to the eight-like solution where the longitudinal motion changes from the fundamental to the second harmonic. For small plasma densities there is only the eighlike solution left.

QUASIPERIODIC WAVES

As disussed in the introduction, for each periodic solution (9)-(11) the initial condition of A_y and φ is properly adjusted (*i.e.* $\varphi(0)' = 0$ and $\varphi(0)$ depends on the laser amplitude a_0) so that no electrostatic oscillations are excited. Quasiperiodic solutions are obtained if the restriction on the initial conditions is dropped. As a consequence, an additional electrostatic oscillation is coupled with the electromagnetic wave. The solution is no longer purely harmonic and the spectrum contains also side bands shifted by the plasma frequency.

For small plasma densities the laser frequency ω differs greatly from the plasma frequency ω_p. As depicted in Fig. 3 the electron trajectory now resembles the periodic vacuum figure-eight motion superposed by the additional electrostatic oscillation. The eight-like trajectory is most clearly seen at the turning points of the longitudinal motion. Due to the nonlinear coupling the amplitude and the frequency of the electromagnetic wave are modulated within a plasma period. These effects are known as amplitude and frequency self-modulation. However, in this density regime no mode conversion occurs, *i.e.* the amplitude of the electrostatic wave is constant.

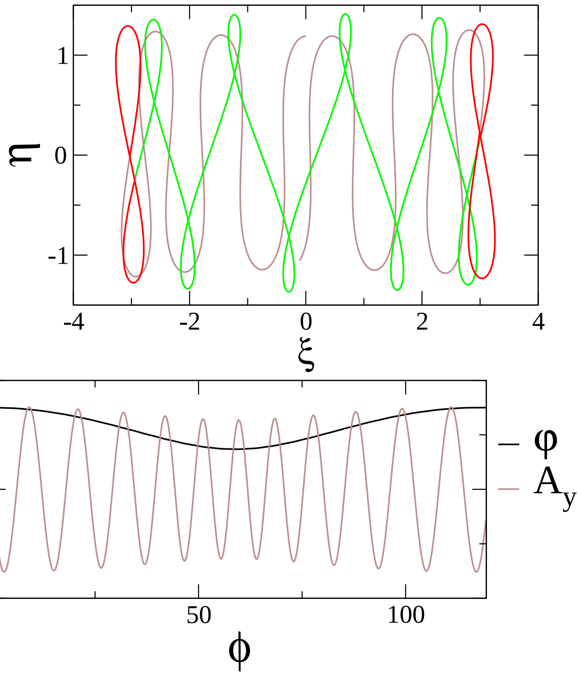

FIG. 3: Quasiperiodic solution for small plasma densities. The eight-like trajectory is coupled with an additional electrostatic oscillation. The amplitude and the frequency of the electromagnetic wave are modulated within a plasma period. ($\epsilon = 0.1$, $\varphi(0) = 3.75$, $A_y'(0) = -1.96$).

For small plasma densities Borovsky *et al.* [26, 30] used a two-time-scale-method to separate the electromagnetic and electrostatic components. Thereby, it is assumed that all quantities depend on a slowly varying phase $s = \epsilon\phi$ and a quickly varying phase f separately. The first one describes the time dependence of the electrostatic oscillation and the last one the time dependence of the electromagnetic wave. The relation between f and s is described by an unkown nonlinear function $\mu(s)$,

$$f(s) = \frac{1}{\epsilon}\int_0^s \mu(s)\, ds. \tag{12}$$

This approach leads in the lowest order to the result

$$A_y(s,f) = -a_0(s)\sin(f) + \epsilon^2\Big(-\frac{3}{256}a_0(s)^3\varphi_0(s)^{-2}\sin(3f) - a_2(s)\sin(f)\Big), \tag{13a}$$
$$\varphi(s,f) = \varphi_0(s) + \epsilon^2\left(\frac{1}{16}a_0(s)^2\varphi_0(s)^{-1}\cos(2f) + \varphi_2(s)\right), \tag{13b}$$
$$\varphi_0(s)_{ss} + \frac{1}{2}\left(1 - \varphi_0(s)^{-2} - \frac{1}{2}g^2\varphi_0(s)^{-3/2}\right) = 0,$$
$$a_0(s) = g\varphi_0(s)^{1/4},$$
$$\mu(s) = \varphi_0(s)^{-1/2},$$

where $a_2(s)$ can be calculated from the elimination of higher order secular terms [26, 30] and $\varphi_2(s)$ is the next order slowly varying contribution to $\varphi(s, f)$. For this solution the initial condition of A'_y and φ can be choosen independently. The solution accounts in the lowest order for the effect of frequency modulation (time dependent electromagnetic frequency $\mu(s)$) and amplitude modulation (slowly varying electromagnetic amplitude $a(s)$). The coupling between the electrostatic and the electromagnetic wave is expressed by the constants g. In all expressions the electrostatic and electromagnetic timescales are strictly separated. The special case of periodic solutions (9) is obtained if one demands $\varphi_0 = const.$ and $\varphi_2(s) = 0$, which yields

$$g = \sqrt{2(\varphi_0^2 - 1)}\varphi_0^{-1/4}, \quad (14a)$$

$$\varphi_0 = \sqrt{1 + \frac{1}{2}a_0^2}. \quad (14b)$$

For quasiperiodic and periodic solutions the expressions for the highest harmonics coincide. The difference is that for the quasiperiodic case the coefficients are slowly time dependent, $\varphi_0 = \varphi_0(s)$.

Near the critical density there exist many different types of quasiperiodic solutions. Due to the strong coupling it is generally not possibel to approximately separate a solution into a periodic wave and an electrostatic oscillation. Furthermore, the modulation is found to be completely different from the modulation for small plasma densities [31]. Now, the laserfrequency ω and the plasma frequency ω_p are nearly equal. The coupling leads for small finite κ to a beating between the frequencies. But even for $\kappa = 0$ the nonlinear coupling can result in an amplitude self-modulation. While the amplitude modulation is large, the frequency and density modulation are quite small.

In a recent work we analyzed the situation of slightly disturbed periodic solutions [31]. This subclass of quasiperiodic solutions results if the initial condition is a small deviation from the initial condition of the exact periodic solutions (10) and (11).

For the eight-like solution the perturbation yields a surprisingly effective mechanism of mode conversion. Thereby, over time the initially small longitudinal component greatly increases and the solution completely changes its form. The almost-transverse wave is periodically converted into an almost-longitudinal wave (Fig. 4). For efficient mode conversion the propagation distance can be adapted to one half of the modulation period.

Nearly periodic circle-like solutions, on the other hand, are stabilized by intrinsic mode coupling. The pertubation induces merely a small amplitude modulation, but the character of the solution is conserved. The trajectory is still circle-like (Fig. 2), but fills out an annulus of finite width.

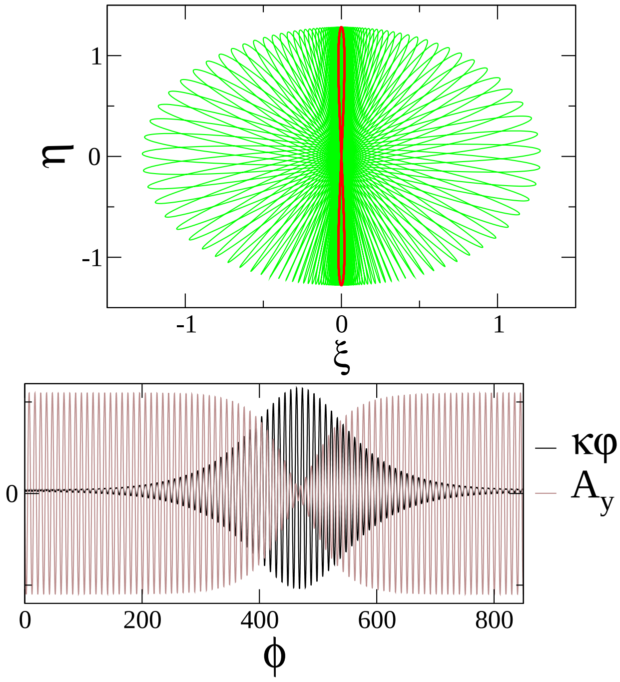

FIG. 4: Quasiperiodic solution for an initially small deviation from the almost-transverse eight-like solution near the critical density. The perturbation results in a large mode conversion. ($\kappa = 1/\epsilon = 0.05$, $\kappa\varphi(0) = 0.116$, $A'_y(0) = -1.96$).

CONCLUSIONS

In conclusion, analytic solutions of relativistic wave propagation in plasmas can be gained by investigation of the Akhiezer-Polovin model. Thereby, the relativistic nonlinearities of high intensity laser matter interaction are completely taken into account. The model has been formulated in different equivalent forms that are suitable for different expansion regimes. Periodic and quasiperiodic waves have been obtained for small and critical plasma densities and fundamental properties have been discussed. Previous results have been generalized and novel solutions for relativistic mode conversion in nearly critical density plasmas have been presented.

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[1] D. Umstadter, Physics of Plasmas **8**, 1774 (2001).
[2] C. E. Max, J. Arons, and A. B. Langdon, Phys. Rev. Lett. **33**, 209 (1974).

[3] X. L. Chen and R. N. Sudan, Phys. Rev. Lett. **70**, 2082 (1993).
[4] R. Lee and M. Lampe, Phys. Rev. Lett. **31**, 1390 (1973).
[5] A. B. Borisov, O. B. Shiryaev, A. McPherson, K. Boyer, and C. K. Rhodes, Plasma Phys. Controlled Fusion **37**, 569 (1995).
[6] B. Dromey, M. Zepf, A. Gopal, K. Lancaster, M. S. Wei, K. Krushelnick, M. Tatarakis, N. Vakakis, S. Moustaizis, R. Kodama, M. Tampo, C. Stoeckl, R. Clarke, H. Habara, D. Neely, S. Karsch, and P. Norreys, Nature Physics **2**, 456 (2006).
[7] T. Tajima and J. M. Dawson, Phys. Rev. Lett. **43**, 267 (1979).
[8] S. P. D. Mangles, C. D. Murphy, Z. Najmudin, A. G. R. Thomas, J. L. Collier, A. E. Dangor, E. J. Divall, P. S. Foster, J. G. Gallacher, C. J. Hooker, D. A. Jaroszynski, A. J. Langley, W. B. Mori, P. A. Norreys, F. S. Tsung, R. Viskup, B. R. Walton, and K. Krushelnick, Nature **431**, 535 (2004).
[9] C. G. R. Geddes, C. Toth, J. van Tilborg, E. Esarey, C. B. Schroeder, D. Bruhwiler, C. Nieter, J. Cary, and W. P. Leemans, Nature **431**, 538 (2004).
[10] J. Faure, Y. Glinec, A. Pukhov, S. Kiselev, S. Gordienko, E. Lefebvre, J.-P. Rousseau, F. Burgy, and V. Malka, Nature **431**, 541 (2004).
[11] T. Brabec and F. Krausz, Rev. Mod. Phys. **72**, 545 (2000).
[12] J. Faure, Y. Glinec, J. J. Santos, F. Ewald, J.-P. Rousseau, S. Kiselev, A. Pukhov, T. Hosokai, and V. Malka, Phys. Rev. Lett. **95**, 205003 (2005).
[13] P. M. Paul, E. S. Toma, P. Breger, G. Mullot, F. Augé, P. Balcou, H. G. Muller, and P. Agostini, Science **292**, 1689 (2001).
[14] Y. Nomura, R. Hörlein, P. Tzallas, B. Dromey, S. Rykovanov, Z. Major, J. Osterhoff, S. Karsch, L. Veisz, M. Zepf, D. Charalambidis, F. Krausz, and G. D. Tsakiris, Nature Physics **5**, 124 (2009).
[15] A. I. Akhiezer and R. V. Polovin, Sov. Phys. JETP **3**, 696 (1956).
[16] P. Kaw and J. Dawson, Phys. Fluids **13**, 472 (1970).
[17] W. Lünow, Plasma Phys. **10**, 879 (1968).
[18] C. Max and F. Perkins, Phys. Rev. Lett. **27**, 1342 (1971).
[19] A. C.-L. Chian and P. C. Clemmow, Journal of Plasma Physics **14**, 505 (1975).
[20] A. Decoster, Phys. Rep. **47**, 285 (1978).
[21] P. Sprangle, E. Esarey, and A. Ting, Phys. Rev. Lett. **64**, 2011 (1990).
[22] W. Mori, C. Decker, and W. Leemans, IEEE Trans. Plasma Sci. **21**, 110 (1993).
[23] A. I. Zhmoginov and G. M. Fraiman, JETP **100**, 895 (2005).
[24] T. C. Pesch and H.-J. Kull, Physics of Plasmas **14**, 3103 (2007).
[25] P. K. Kaw, A. Sen, and E. J. Valeo, Physica D: Nonlinear Phenom. **9**, 96 (1983).
[26] A. V. Borovsky, A. L. Galkin, V. V. Korobkin, and O. B. Shiryaev, Phys. Rev. E **59**, 2253 (1999).
[27] C. E. Max, Phys. Fluids **16**, 1277 (1973).
[28] P. Sprangle, E. Esarey, and A. Ting, Phys. Rev. A **41**, 4463 (1990).
[29] E. Esarey, A. Ting, P. Sprangle, D. Umstadter, and X. Liu, IEEE Trans. Plasma Sci. **21**, 95 (1993).
[30] O. B. Shiryaev, Physics of Plasmas **15**, 012308 (2008).
[31] T. C. Pesch and H.-J. Kull, EPL **85**, 25003 (2009).
[32] B. B. Winkles and O. Eldridge, Phys. Fluids **15**, 1790 (1972).
[33] P. C. Clemmow, Journal of Plasma Physics **12**, 297 (1974).