# Dispersion relations of nonlinearly coupled electromagnetic and electrostatic waves in relativistic plasmas

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(Received 15 September 2009; accepted 22 December 2009; published online 19 January 2010)

In the present work large amplitude electromagnetic waves in cold plasmas at relativistic intensities are studied analytically in a plane wave geometry. Special attention is drawn to the nonlinear coupling of electromagnetic and electrostatic modes. In the framework of the Akhiezer–Polovin model, periodic and more general quasiperiodic waves are taken into account. For small densities a two-time-scale method is used to calculate an analytical solution up to the fourth order in the plasma density. Nonlinear dispersion relations are calculated for coupled waves, taking into account the full plasma response for linear as well as for circular polarization. In the presence of a large amplitude electrostatic wave, the results show a major difference from the commonly considered dispersion relation for electromagnetic waves. Finally, the solutions of the Akhiezer–Polovin model are compared with particle-in-cell simulations. © *2010 American Institute of Physics*.
[doi:10.1063/1.3292648]

## I. INTRODUCTION

In the past decade the interaction between laser pulses and matter at relativistic light intensities has become a field of growing interest. The introduction of modern laser systems with intensities above the relativistic intensity threshold of roughly $10^{18}$ W/cm$^2$ opened the door to a regime in which electrons at the laser focus oscillate at relativistic velocities.[1,2] The relativistic nonlinearities in the resulting laser-plasma interaction and the violation of the plasma local charge neutrality lead to a whole bunch of new phenomena, such as self-focusing and self-modulation,[3,4] filamentation,[5,6] soliton generation,[7–10] and particle acceleration[11–14] as well as pulse compression and attosecond pulse generation.[15–18]

In this work the problem of stationary plane ultraintense electromagnetic waves in cold tenuous plasmas is studied in the framework of the Akhiezer–Polovin model.[19] This model is completely relativistic and allows one to study the propagation properties of waves from small up to ultralarge intensities in detail. Although it is a plane wave model, it can also approximately be used for the description of sufficiently long pulses. The basic model equations follow from Maxwell equations and the relativistic equations of motion under the following assumptions. The electrons are treated relativistically, the ions are assumed to be stationary, and thermal effects are neglected. For plane waves with wavenumber $k$ and a constant phase velocity $v_{\mathrm{ph}}=\omega/k$ propagating in the $x$-direction, all quantities depend only on the phase $\hat{\phi}=\omega(t-x/v_{\mathrm{ph}})$. It allows one to transform the partial differential equations into ordinary differential equations. The resulting waves are stationary waves and are of great interest for the understanding of stable long distance propagation in plasmas. Of special interest is the nonlinear coupling between transverse electromagnetic and longitudinal electrostatic modes. This coupling yields either periodic or quasiperiodic waves.

The Akhiezer–Polovin model can be expressed in various equivalent forms by transforming to different variables. In the original work Akhiezer and Polovin used the electron momenta $\boldsymbol{p}$ as independent variables.[19,20] A simpler system of equations for the Lagrangian coordinates $\eta,\zeta,\xi$ of the particles (as a function of proper time) was introduced in Ref. 21. However, for the investigation of quasiperiodic solutions, it is convenient to use another set of equations for the potentials $\boldsymbol{A}$ and $\varphi$ as variables[22–26] since the separation of the solution by the two-time-scale method is more distinct in $\varphi$ than $\xi$ or $p_x$.

Periodic waves result if the longitudinal component oscillates with a multiple of the laser frequency $\omega$. A well-known periodic solution is the figure-eight trajectory of a single relativistic electron in an electromagnetic wave. Most previous work on the Akhiezer–Polovin model focused on periodic waves.[19–22,27–32] However, periodic waves exist only for properly adjusted initial conditions. For general initial conditions an additional electrostatic oscillation is excited. Physically, such an oscillation is also induced by the relativistic light pressure (or the Lorentz force) of the laser beam (wake field generation). The characteristic frequency of the electrostatic oscillation is the plasma frequency $\omega_{\mathrm{p}}$. In general, the result of the nonlinear coupling of the electromagnetic wave to this electrostatic oscillation is a quasiperiodic wave. The electron orbits are in general nonclosed.[23,33] Due to the nonlinear coupling through the density $n$ and the relativistic gamma factor $\gamma$, the amplitude and the frequency of the electromagnetic wave are modulated within a plasma period. Physically, the amplitude and frequency self-modulation can be explained as follows.[23] The electrostatic oscillation provokes an additional density modulation in comparison with the periodic case. This density modulation induces a frequency modulation of the electromagnetic wave, $\hat{\Omega}\sim\sqrt{n/\gamma}$. The frequency modulation, in turn, yields

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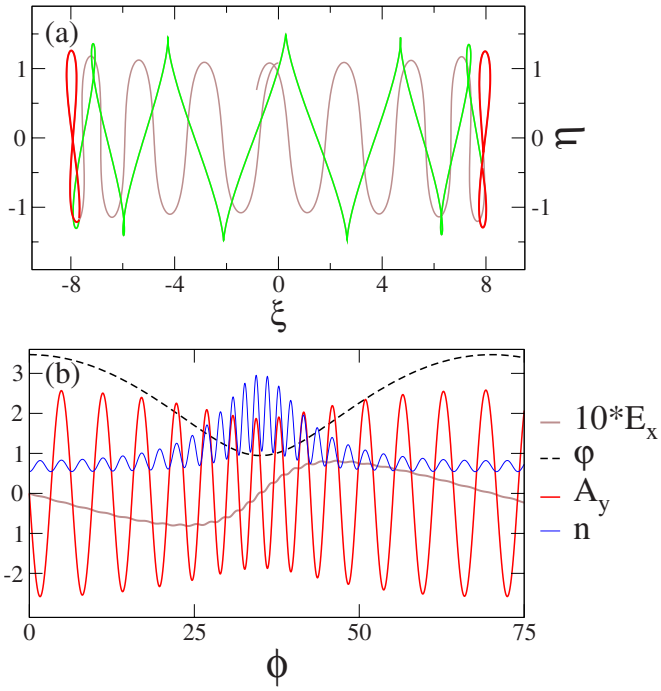


FIG. 1. (Color online) Nonlinearly coupled electromagnetic and electrostatic wave for small plasma densities (numerical solution of the Akhiezer–Polovin model). (a) The eightlike trajectory, visible at the turning points, is coupled with an additional electrostatic oscillation. (b) Due to the nonlinearity of the coupling, the amplitude and the frequency of the electromagnetic wave are modulated within a plasma period [$\epsilon=0.08$, $\varphi(0)=3.46$, and $A_y'(0)=-1.38$ for linear polarization].

the modulation of the amplitude $a$ because of the adiabatic invariant $I=E/\hat{\Omega}\propto a^2\hat{\Omega}$ of an oscillator with the energy $E$ and with a slowly time varying frequency $\hat{\Omega}$. The whole modulation behavior is depicted in Fig. 1 for a typical quasiperiodic solution of the Akhiezer–Polovin model. The density bunching yields an increase in the frequency but a decrease in the amplitude of the electromagnetic wave. The electron trajectory resembles the periodic vacuum figure-eight motion superposed by the additional electrostatic oscillation. The eightlike trajectory is most clearly seen at the turning points of the longitudinal motion.

Besides the small density regime the critical density regime receives growing interest. In recent work an effective mechanism of nonlinear mode conversion was discovered for slightly disturbed periodic solutions.[24] In certain situations these solutions have the structure of envelope solitons.[25]

In this work a consistent solution for quasiperiodic waves is presented for small densities. The Akhiezer–Polovin model is solved for linear as well as for circular polarization using a two-time-scale method. Thereby, previous results of Borovsky *et al.*[23] are generalized. Especially, a simplified elimination of secular terms allows to calculate consistent higher-order solutions. A general method for the derivation of the dispersion relation for coupled electromagnetic and electrostatic waves is introduced. Analytical expressions are derived in the limiting cases of small and large amplitude electrostatic waves. The general case of the coupling to an arbitrarily large amplitude electrostatic wave is investigated numerically. This allows us to tackle the long-standing problem of the correct relativistic modification to the plasma frequency in the dispersion relation for nonlinearly coupled electromagnetic and electrostatic waves. As a major result the correction factor can be limited to a narrow band of values bounded by the analytical solutions. Finally, the solutions of the Akhiezer–Polovin model are compared with particle-in-cell (PIC) simulations.

In the following dimensionless quantities are used. They are connected with the dimensional quantities (dashed) by $t=\omega t'$, $x=x'\omega/c$, $p=p'/(mc)$, $A=eA'/(mc^2)$, and $n=n'/n_0$, where $n_0$ is the density of the ions at rest. For clarity, the dimensional quantities are reintroduced in the dispersion relations. All equations written in dimensional form are indicated $(*)$.

## II. SMALL DENSITY REGIME

The basic set of equations of the Akhiezer–Polovin model [e.g., Eqs. (A12a) and (A12b) in the Appendix] contains the dimensionless parameter

$$\alpha=\frac{\omega_p^2 v_{ph}^2}{v_{ph}^2-1}. \tag{1}$$

This parameter determines the phase velocity of the wave for a given plasma density in the form

$$v_{ph}=\frac{1}{\sqrt{1-\frac{\omega_p^2}{\alpha}}}. \tag{2}$$

For each type of wave the value of $\alpha$ has to be calculated. Afterward the phase velocity of this wave can be obtained by Eq. (2). Only in the linear regime $\alpha$ is equal to one. For nonlinear waves $\alpha$ is a function of frequency and wave amplitudes. If no additional electrostatic wave is present (periodic solutions), $\alpha$ is in the lowest order equal to the mean relativistic gamma factor of the particles.[19] One main goal of this work is the calculation of $\alpha$ for coupled electromagnetic and electrostatic waves (quasiperiodic solutions).

Using $v_{ph}=\omega/k$ $(*)$, Eq. (2) is equivalent to the dispersion relation

$$c^2k^2=\omega^2-\frac{\omega_p^2}{\alpha} \quad (*). \tag{3}$$

In this form $\alpha$ determines the wavelength for a given frequency and plasma density. The nonlinear correction $\alpha$ appears in Eqs. (2) and (3) as a (dimensionless) correction factor to the plasma frequency. In this sense $\omega_p/\sqrt{\alpha}$ can be viewed as an effective plasma frequency.

Furthermore, following Borovsky *et al.*[23] we introduce the parameter

$$\epsilon=\sqrt{v_{ph}^2-1}. \tag{4}$$

For small plasma densities the phase velocity $v_{ph}$ is slightly greater than one so that in this situation $\epsilon$ can be used as a small parameter for a perturbation expansion.

Using as the independent parameter, the phase $\phi=\sqrt{\alpha}\hat{\phi}$, the basic set of equations for the potentials becomes

$$A'' + FA = 0, \tag{5a}$$

$$\varphi'' + (F\varphi - 1) = 0, \tag{5b}$$

$$F = \frac{n}{\gamma} = \sqrt{\frac{1+\epsilon^2}{\varphi^2 + \epsilon^2(1+A^2)}}. \tag{5c}$$

An detailed derivation of Eqs. (5a)–(5c) can be found in the Appendix. These equations are equivalent to the equations of Akhiezer and Polovin and describe the full relativistic coupling between the transverse electromagnetic and the longitudinal electrostatic oscillations. It is interesting to note that in Eq. (5b), the coupling of the electrostatic oscillation to the electromagnetic one is given by the substitution $1 \to 1+A^2$ in the denominator of $F$. From Eqs. (5a) and (5b) follows the conservation of the total energy

$$\tfrac{1}{2}(\varphi'^2 + \epsilon^2 A'^2) + \epsilon^2\gamma = \text{constant}, \tag{6}$$

where the first part is the field energy and $\gamma = \sqrt{1+p^2}$ is the relativistic gamma factor (particle energy). A discussion of all invariants of Eqs. (5a)–(5c) can be found in Ref. 23.

From the potentials one can deduce all other quantities such as the momenta $p_x$ ($p_y = A_y$), the particle energy $\gamma$, the density $n$, and the electric and magnetic fields, e.g., (Appendix)

$$p_x = \frac{1}{\epsilon^2}\left(\sqrt{\varphi^2 + \epsilon^2(1+A^2)} - \sqrt{1+\epsilon^2}\varphi\right), \tag{7a}$$

$$\gamma = \frac{1}{\epsilon^2}\left(\sqrt{1+\epsilon^2}\sqrt{\varphi^2 + \epsilon^2(1+A^2)} - \varphi\right), \tag{7b}$$

$$n = 1 + \frac{1}{\epsilon^2}(1 - F\varphi), \tag{7c}$$

$$E_x = \varphi'\sqrt{\alpha}/\sqrt{1+\epsilon^2}, \quad B_z = 0, \tag{7d}$$

$$E_y = -A_y'\sqrt{\alpha}, \quad B_z = E_y/\sqrt{1+\epsilon^2}. \tag{7e}$$

In the subsequent analysis we use a two-time-scale method to separate electromagnetic and electrostatic components and solve the equations up to order $\epsilon^4$. In this framework we assume that all quantities depend on a slowly varying phase $s = \epsilon\phi$ and a quickly varying phase $f$ separately. The first one describes the time dependence of the electrostatic oscillation, and the last one describes the time dependence of the electromagnetic wave. The relation between $f$ and $s$ is described by an unknown nonlinear function $\Omega(s) > 0$ (time dependent frequency of the electromagnetic wave),

$$f(s) = \frac{1}{\epsilon}\int_0^s \Omega(s)ds. \tag{8}$$

The introduction of $\Omega(s)$ allows one to set the phase of the occurring trigonometric functions equal to $f$. The derivative with respect to the phase $\phi$ can now be expressed by partial derivatives with respect to $f$ and $s$ (indicated by the indices $f$ and $s$),

$$\frac{d}{d\phi} = \epsilon\frac{d}{ds} = \Omega\partial_f + \epsilon\partial_s, \tag{9a}$$

$$\frac{d^2}{d\phi^2} = \Omega^2\partial_{ff} + \epsilon(2\Omega\partial_{fs} + \Omega_s\partial_f) + \epsilon^2\partial_{ss}. \tag{9b}$$

Furthermore, it is assumed that the potentials can be expressed as a series in $\epsilon$ for small plasma densities,

$$A(s,f) = A_0(s,f) + \epsilon A_1(s,f) + \epsilon^2 A_2(s,f) + \ldots, \tag{10a}$$

$$\varphi(s,f) = \varphi_0(s,f) + \epsilon\varphi_1(s,f) + \epsilon^2\varphi_2(s,f) + \ldots. \tag{10b}$$

In contrast to Borovsky *et al.*,[23] it is also assumed that $\Omega(s)$ is a series in $\epsilon$,

$$\Omega(s) = \Omega_0(s) + \epsilon\Omega_1(s) + \epsilon^2\Omega_2(s) + \ldots. \tag{11}$$

This considerably simplifies the elimination of secular terms in higher orders and allows one to obtain a consistent solution of arbitrary order in $\epsilon$.

With this ansatz we solve the basic set of Eqs. (5a)–(5c) order by order in $\epsilon$. Together with the condition of consistency (elimination of secular terms), there are sufficient equations to calculate $\varphi_n(s,f)$, $A_n(s,f)$, and $\Omega_n(s)$ in every order.

## III. LINEAR POLARIZATION

The initial condition for linear polarization is chosen as $A = A_y = 0$ at $\phi = 0$ without loss of generality since $\phi$ is only determined up to a constant. For simplicity the lowest order solution is chosen to satisfy the initial condition. Therefore, $\varphi_n$, $\varphi_{n,s}$, and $A_{n,f}$ are set equal to zero at $\phi = 0$ for $n \geq 1$.

The lowest order result for $\varphi$ is

$$\varphi_0(s,f) = \varphi_0(s) > 0, \tag{12a}$$

$$\varphi_0(s)_{ss} + \tfrac{1}{2}\left(1 - \varphi_0^{-2} - \tfrac{1}{2}g_0^2\varphi_0^{-3/2}\right) = 0. \tag{12b}$$

The differential equation can be integrated once,

$$\varphi_0(s)_s^2 + \underbrace{\left(\varphi_0 + \varphi_0^{-1} + g_0^2\varphi_0^{-1/2}\right)}_{V(\varphi_0)} = \text{constant} = E_0. \tag{13}$$

In the context of mechanics this equation corresponds to the energy conservation for the motion of a particle with energy $E_0$ in a potential $V(\varphi_0)$. Our calculation finally yields the fully consistent solution up to order $\epsilon^4$,

$$\varphi_1(s,f) = 0, \tag{14a}$$

$$\varphi_2(s,f) = \tfrac{1}{16}g_0^2\varphi_0^{-1/2}\cos(2f) + \varphi_{2R}(s), \tag{14b}$$

$$\begin{aligned}\varphi_{2R}(s)_{ss} &+ \tfrac{\varphi_{2R}}{\varphi_0^3}\left(1 + \tfrac{3}{8}g_0^2\varphi_0^{1/2}\right) - \tfrac{1}{2}g_0g_2\varphi_0^{-3/2} \\ &- \tfrac{1}{8}\Big(1 + g_0^2\varphi_0^{-3/2} + 2\varphi_0^{-2} + \tfrac{3}{16}g_0^2E_0\varphi_0^{-5/2} \\ &- \tfrac{15}{16}g_0^4\varphi_0^{-3} - \tfrac{45}{16}g_0^2\varphi_0^{-7/2} - 3\varphi_0^{-4}\Big) = 0,\end{aligned} \tag{14c}$$

$$\varphi_3(s,f) = \frac{3}{64}g_0^2\frac{\varphi_{0s}}{\varphi_0}\sin(2f), \tag{14d}$$

$$\varphi_4(s,f) = \tfrac{19}{2^{13}} g_0^4 \varphi_0^{-2} \cos(4f) + \tfrac{1}{8} g_0 \times \Big[\tfrac{1}{16} g_0 \big(3\varphi_0^{-1/2} - \tfrac{15}{4} E_0 \varphi_0^{-3/2} + \tfrac{49}{16} g_0^2 \varphi_0^{-2} + \tfrac{3}{4} \varphi_0^{-5/2}\big) - \tfrac{1}{4} g_0 \varphi_{2R} \varphi_0^{-3/2} + g_2 \varphi_0^{-1/2}\Big] \cos(2f) + \varphi_{4R}(s), \tag{14e}$$

$$A_0(s,f) = -a_0(s)\sin(f), \tag{15a}$$

$$a_0(s) = g_0 \varphi_0^{1/4}, \tag{15b}$$

$$A_1(s,f) = 0, \tag{15c}$$

$$A_2(s,f) = -\tfrac{3}{256} g_0^3 \varphi_0^{-5/4} \sin(3f) - a_2(s)\sin(f), \tag{15d}$$

$$a_2(s) = \tfrac{1}{64} g_0 (3E_0 \varphi_0^{-3/4} + g_0^2 \varphi_0^{-5/4} + 3\varphi_0^{-7/4}) + \tfrac{1}{4} g_0 \varphi_{2R} \varphi_0^{-3/4} + g_2 \varphi_0^{1/4}, \tag{15e}$$

$$A_3(s,f) = \tfrac{21}{2048} g_0^3 \tfrac{\varphi_{0s}}{\varphi_0^{7/4}} \cos(3f) - a_3(s)\sin(f), \tag{15f}$$

$$A_4(s,f) = -\tfrac{85}{2^{17}} g_0^5 \varphi_0^{-11/4} \sin(5f) - a_{43}(s)\sin(3f) - a_{41}(s)\sin(f), \tag{15g}$$

$$\Omega_0(s) = \sqrt{\frac{1}{\varphi_0}}, \tag{16a}$$

$$\Omega_1(s) = 0, \tag{16b}$$

$$2\frac{\Omega_2(s)}{\Omega_0} = -\frac{\varphi_{2R}}{\varphi_0} + \frac{9}{16} - \frac{3}{16}\varphi_0^{-2} - \frac{3}{16} E_0 \varphi_0^{-1} - \frac{3}{32} g_0^2 \varphi_0^{-3/2}, \tag{16c}$$

$$\Omega_3(s) = 0. \tag{16d}$$

This result extends the results in Ref. 23 by three and the results in Refs. 26 and 34 by two orders in $\epsilon$. Furthermore, the prefactors of the higher harmonics of the order $\epsilon^2$ differ from the expressions calculated in Ref. 34 but coincide with the one from Ref. 26. In addition a purely slowly varying part $\varphi_{2R}(s)$ is found in $\varphi_2$. The corrections introduced in Refs. 26 and 34 to account for the elimination of secular growth are considerably simplified due to the ansatz of $\Omega(s)$ as a series in $\epsilon$ [Eq. (11)]. This simplification is the key to the calculation of higher orders. The expansion up to order $\epsilon^4$ is necessary to obtain the equation of motion for $\varphi_{2R}(s)$ [Eq. (14c)]. This is the lowest nontrivial correction to the electrostatic oscillation $\varphi_0(s)$. The correction is important to describe the numerical result for $\varphi(s)$ with good accuracy. Without $\varphi_{2R}(s)$ the numerical and the analytical solutions of $\varphi$ get out of phase within few cycles (Fig. 2). For the same reason $a_2(s)$ is necessary to describe the behavior of $A$ properly. Another reason for a higher-order expansion is that some features of the higher-order solution remain valid up to larger plasma densities. The first and third harmonics in $A_4$ as well as the first harmonic in $A_3$ can only be calculated in even higher orders.

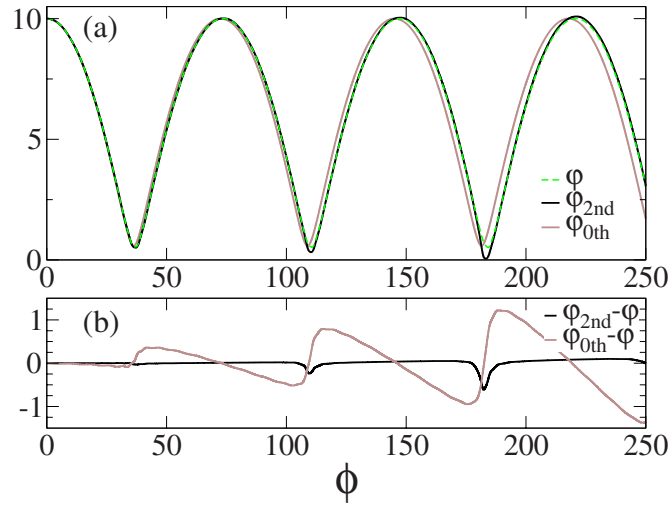


FIG. 2. (Color online) (a) Comparison of the results of order $\epsilon^0$ ($\varphi_{0\text{th}}=\varphi_0$) and order $\epsilon^2$ ($\varphi_{2\text{nd}}=\varphi_0+\epsilon^2\varphi_2$) with the numerical solution for the scalar potential $\varphi$. The higher-order solution remains even for large times in phase with the numerical solution. (b) For better visual convenience the differences between the analytical solutions and the numerical solution are plotted [$\epsilon=0.2$, $A'(0)=1.6$, and $\varphi(0)=10.0$].

The solutions (12a)–(16d) completely account for the effect of frequency modulation [time dependent electromagnetic frequency $\Omega(s)$] and amplitude modulation [$a(s)$]. In all expressions the electrostatic and electromagnetic time scales are strictly separated. Only in the lowest order the adiabatic invariant $a^2(s)\Omega(s)$ is an exact invariant. The coupling between the electrostatic and the electromagnetic part is expressed by the integration constants $g_i$. The value of $g_i$ can be calculated by inserting the initial condition mentioned above [this yields, for example, $g_0=A(0)/\varphi(0)$]. Due to relativistic corrections and the coupling to the electromagnetic wave, the electrostatic wave oscillates with the modified plasma wave frequency $\Omega_p=2\pi/T_s$, where $T_s=\oint_\varphi ds$ instead of $\omega_p$. Generally, the modified plasma wave frequency $\Omega_p$ differs from the effective plasma frequency $\omega_p/\sqrt{\alpha}$ defined by the dispersion relation (3).

The special case of periodic solutions is included in Eqs. (12a)–(16d) if one demands $\varphi_0$=constant and $\varphi_{nR}=0$ for $n\geq 1$. This yields

$$\varphi_0 = \sqrt{1 + \tfrac{1}{2} a_0^2}, \tag{17a}$$

$$g_0 = \sqrt{2(\varphi_0^2 - 1)}\,\varphi_0^{-1/4}, \tag{17b}$$

$$g_2 = -\tfrac{3}{64}(1 + 2\varphi_0^{-2}). \tag{17c}$$

Furthermore, for periodic solutions it is $\langle\gamma\rangle=\varphi_0$ and $\Omega=\varphi_0^{-1/2}$, $f=\Omega\phi$. For quasiperiodic and periodic solutions, the highest harmonics expressions coincide in each order. The difference is that for the quasiperiodic case, the coefficients are slowly time dependent, $\varphi_0=\varphi_0(s)$. However, in the spectra of the quasiperiodic solution, there dominate the slowly varying contributions with peaks at multiples of the plasma wave frequency $\Omega_p$.

A rarely mentioned intrinsic problem of the Akhiezer–Polovin model is the derivation of the dispersion relation,

i.e., the relation between $v_{\rm ph}$ and $\omega_{\rm p}$. From Eqs. (2) and (3) it follows that the dispersion relation directly depends on the factor $\alpha$ ($\sqrt{\alpha}$ is the relativistic correction to the plasma frequency). However, with the scaling of the phase $\alpha$ completely disappears from the basic equations (5a)–(5c) and is only present in the dependency of the solutions on the laboratory phase $\hat{\phi}$. Therefore, $\alpha$ and the dispersion relation cannot be calculated from the Akhiezer–Polovin model without any additional assumption on this dependency. For the special case of periodic solutions, it is possible to determine $\alpha$ by the condition that the solutions are $2\pi$-periodic functions in the phase $\hat{\phi}$.[19,21] This is equivalent to the assumption that the laboratory frequency $\hat{\Omega}=\Omega\sqrt{\alpha}$ of the wave in the plasma corresponds to the laser frequency $\omega$. For quasiperiodic solutions this condition has to be generalized. One possibility is to consider the plasma boundary where the laser beam enters and the $2\pi$-periodicity condition is still valid. Due to the stationarity of the Akhiezer–Polovin model, this approach is beyond the scope of the present work. Instead, one can use the approximation that the laser frequency is equal to the frequency $\Omega\sqrt{\alpha}$ averaged over the electrostatic oscillation time scale $T_s$,

$$\langle\Omega\rangle = \frac{1}{T_s}\oint_{\varphi}\Omega ds = \frac{\omega}{\sqrt{\alpha}}. \tag{18}$$

This is in agreement with the fact that in an inhomogeneous plasma (in average), the wavelength of an electromagnetic wave changes, but the frequency remains constant.

In the lowest order ($ds=d\varphi_0/\sqrt{E-V(\varphi_0)}$), this approach yields

$$\alpha_0 = \left(\frac{\oint \frac{d\varphi_0}{\sqrt{E-V(\varphi_0)}}}{\oint \frac{d\varphi_0}{\sqrt{\varphi_0[E-V(\varphi_0)]}}}\right)^2, \tag{19}$$

taking $\alpha$ itself to be a series in $\epsilon$. In the same way further corrections to $\alpha$ can be calculated using the higher-order results of Eqs. (14a)–(16d). The approximate phase velocity and dispersion relation can then be calculated from Eqs. (2) and (3), yielding

$$v_{\rm ph}^2 - 1 = \epsilon^2 = \frac{\omega_{\rm p}^2}{\alpha_0} + \left(1-\frac{\alpha_2}{\alpha_0}\right)\frac{\omega_{\rm p}^4}{\alpha_0^2\omega^2} + \mathcal{O}(\omega_{\rm p}^6), \tag{20}$$

and up to order $\omega_{\rm p}^4$,

$$c^2k^2 = \omega^2 - \frac{\omega_{\rm p}^2}{\alpha_0} + \frac{\alpha_2}{\alpha_0}\frac{\omega_{\rm p}^4}{\alpha_0^2\omega^2} \quad (*), \tag{21}$$

independently of the value of $\alpha_1$. A quantization condition as in that in Ref. 23, which is restricted to the case where $\Omega$ is a multiple of $\Omega_{\rm p}$, is not required by this approach.

For periodic solutions, where $\alpha_0=\langle\gamma\rangle$ and $\alpha_2=3/32a_0^2\varphi_0^{-1}$, Eq. (21) yields the dispersion relation Eq. (22) of Ref. 21. This already shows that $\alpha$ deviates from $\langle\gamma\rangle$ in higher orders in the plasma density. For quasiperiodic solutions even $\alpha_0$ is generally not equal to $\langle\gamma\rangle$. However, in many works $\alpha_0$ is approximately set equal to $\langle\gamma\rangle$ even if a coupling to an additional electrostatic wave is involved. In the following, $\alpha_0$ is calculated analytically and numerically for quasiperiodic solutions, and the dependence of $\alpha_0$ on the amplitude of the electrostatic wave is discussed.

## IV. COUPLING TO A SMALL AMPLITUDE PLASMA WAVE

If the additional electrostatic wave $\varphi_0(s)$ has a small or ultralarge amplitude, Eqs. (12a)–(21) can be further analytically investigated. In the first case, it is assumed that the solution is in the lowest order a periodic electromagnetic wave of arbitrarily large amplitude, which is accompanied by a small electrostatic wave. Then, one has $\varphi_0(s)=\varphi_{\rm p}+\varphi_r(s)$, and $\varphi_{\rm p}=(1+a_0^2/2)^{1/2}$ obeys Eqs. (17a) for periodic solutions. For the initial condition $\varphi'(0)=0$, Eq. (12b) leads in linear order in $\varphi_r(s)$ to

$$\varphi_0(\phi) = \varphi_{\rm p} + B_0\cos(\Omega_{\rm p}\phi), \quad \Omega_{\rm p} = \frac{1}{2}\sqrt{\frac{3\varphi_{\rm p}^2+1}{\varphi_{\rm p}^3}}\epsilon. \tag{22}$$

With Eq. (15b), Eq. (16a), and $\Omega_{\rm EM}=\varphi_{\rm p}^{-1/2}$, the amplitude and frequency modulation of the electromagnetic wave are

$$a_0(\phi) = a_0 + \tfrac{1}{4}g\varphi_{\rm p}^{-3/4}B_0\cos(\Omega_{\rm p}\phi), \tag{23a}$$

$$\Omega(\phi) = \Omega_{\rm EM} - \tfrac{1}{2}\varphi_{\rm p}^{-3/2}B_0\cos(\Omega_{\rm p}\phi), \tag{23b}$$

$$f(\phi) = \Omega_{\rm EM}\phi - (3\varphi_{\rm p}^2+1)^{-1/2}\epsilon^{-1}B_0\sin(\Omega_{\rm p}\phi), \tag{23c}$$

so that in linear order

$$A_0(\phi) = -a_0\sin(\Omega_{\rm EM}\phi) - B_+\sin[(\Omega_{\rm EM}+\Omega_{\rm p})\phi] - B_-\sin[(\Omega_{\rm EM}-\Omega_{\rm p})\phi], \tag{24a}$$

$$B_\pm = \tfrac{1}{2}a_0B_0\left[\tfrac{1}{4}\varphi_{\rm p}^{-1} \mp (3\varphi_{\rm p}^2+1)^{-1/2}\epsilon^{-1}\right]. \tag{24b}$$

The frequency spectrum of $A_0$ contains sidebands at $\Omega_{\rm EM}\pm\Omega_{\rm p}$. However, the contribution from $\Omega_{\rm EM}+\Omega_{\rm p}$ is smaller than that from $\Omega_{\rm EM}-\Omega_{\rm p}$. The last result is valid in the scope of perturbation theory, i.e., as long as $B_\pm$ is not too large due to the factor $\epsilon^{-1}$. For an initial condition $\varphi'(0)\neq 0$, there are cos-terms of the same structure. In a similar way the contribution of $\varphi_{\rm p}$ to the higher-order harmonics of $A$ and $\varphi$ can be calculated.

The calculation of $\alpha_0$ according to Eq. (18) yields

$$\alpha_0 = \varphi_{\rm p}. \tag{25}$$

In addition, from Eq. (7b) one finds that again $\langle\gamma\rangle=\varphi_{\rm p}$ if the average is performed over large times. Therefore, the dispersion relation can up to order $\omega_{\rm p}^2$ still be written as for periodic solutions,

$$c^2k^2 = \omega^2 - \frac{\omega_{\rm p}^2}{\langle\gamma\rangle} \quad (*). \tag{26}$$

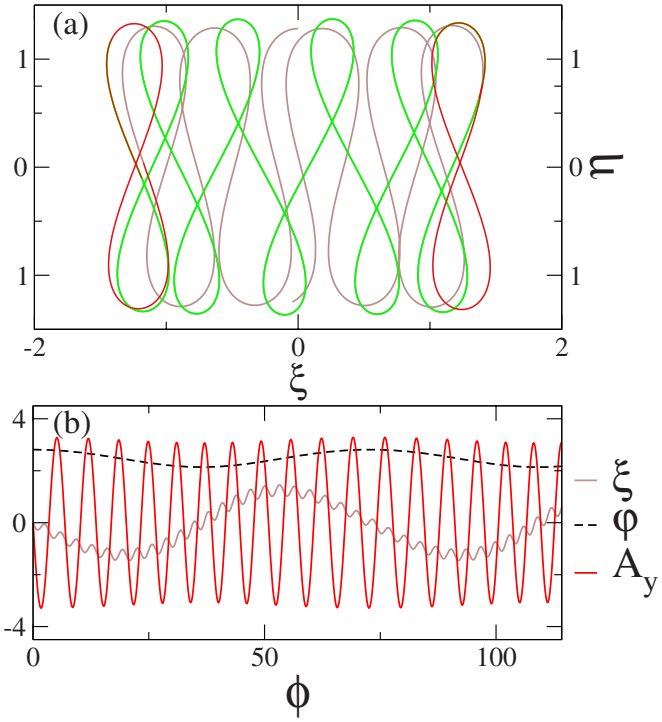


FIG. 3. (Color online) Coupling of a small amplitude electrostatic oscillation to an electromagnetic wave [$\epsilon=0.1$, $\varphi(0)=2.8$ for $\varphi_p=\alpha=2.35$, $A_y'(0)=-1.95$]. The analytically calculated period of the electrostatic oscillation is $\hat{T}=2\pi/\hat{\Omega}_p=70.8$ and coincides well with the numerical results. Due to the nonlinear coupling with the electromagnetic wave, the electrostatic frequency $\hat{\Omega}_p=0.089$ deviates from the plasma frequency $\omega_p=0.15$ as well as from effective plasma frequency $\omega_p/\sqrt{\langle\gamma\rangle}=0.1$.

The laboratory frame quantities can be calculated by

$$\hat{\Omega}=\Omega\sqrt{\alpha},\quad \hat{\phi}=\phi/\sqrt{\alpha},\quad \Omega\phi=\hat{\Omega}\hat{\phi}. \tag{27}$$

Furthermore, for small densities it is $\epsilon=\omega_p/\sqrt{\alpha}$. Therefore, the plasma wave frequency in the laboratory frame is

$$\hat{\Omega}_p=\sqrt{\alpha}\Omega_p=\frac{1}{2}\sqrt{\frac{3\varphi_p^2+1}{\varphi_p^3}}\omega_p. \tag{28}$$

The numerical result for the coupling of a small amplitude electrostatic oscillation to an electromagnetic wave is illustrated in Fig. 3. The electrostatic wave oscillates with the plasma wave frequency $\hat{\Omega}_p$ [Eq. (28)]. This frequency is not equal to the effective plasma frequency $\varphi_p^{-1/2}\omega_p$, which appears in the dispersion relation (26). The latter one corresponds to the well-known relativistic mass increase modification of the plasma frequency due to the large amplitude electromagnetic wave. The difference in $\hat{\Omega}_p$ results from the consistent treatment of the nonlinear coupling between the electromagnetic and electrostatic wave, which was apparently not considered in a previous work.

The calculation above for the results of order $B_0$ can easily be generalized to weakly relativistic plasma oscillations by taking into account terms up to order $B_0^3$. This yields further corrections to the plasma wave frequency as well as to the dispersion relation of order $B_0^2$,

$$\begin{aligned}\varphi_0=\varphi_p+B_0\cos(\hat{\Omega}_p\hat{\phi})+\frac{1}{4}B_0^2\frac{1}{2\varphi_p}\frac{5\varphi_p^2+3}{3\varphi_p^2+1}\\ \times[3-\cos(2\hat{\Omega}_p\hat{\phi})]\\ +\frac{7}{64}B_0^3\frac{1}{7\varphi_p^2}\frac{45\varphi_p^4+53\varphi_p^2+14}{9\varphi_p^4+6\varphi_p^2+1}\cos(3\hat{\Omega}_p\hat{\phi}),\end{aligned} \tag{29a}$$

$$\begin{aligned}\hat{\Omega}_p=\frac{1}{2}\sqrt{\frac{3\varphi_p^2+1}{\varphi_p^3}}\omega_p\\ -\frac{3}{16}B_0^2\sqrt{\frac{\varphi_p^3}{3\varphi_p^2+1}}\frac{1}{2\varphi_p^3}\frac{5\varphi_p^4+7\varphi_p^2+4}{3\varphi_p^4+\varphi_p^2}\omega_p,\end{aligned} \tag{29b}$$

$$c^2k^2=\omega^2-\left(1-\frac{3}{8}B_0^2\frac{2}{\varphi_p^2}\frac{\varphi_p^2+1}{3\varphi_p^2+1}\right)\frac{\omega_p^2}{\varphi_p}\quad(*). \tag{29c}$$

Now, the dispersion relation differs from the periodic solution. The effective plasma frequency in the dispersion relation is slightly reduced due to the coupling to the small amplitude plasma wave.

## V. COUPLING TO AN ULTRALARGE AMPLITUDE PLASMA WAVE

In the second case we consider the coupling of the electromagnetic wave to an ultralarge amplitude plasma wave so that the amplitude of the plasma wave is large compared to the electromagnetic amplitude. Nevertheless, for a sufficiently large plasma wave the electromagnetic wave can still be relativistic. In this situation Eq. (12b) reduces in a parabolic approximation to

$$\varphi_0(s)_{ss}+\tfrac{1}{2}=0. \tag{30}$$

For $\varphi(0)\gg 1$ and the initial condition $\varphi'(0)=0$, the solution is in the lowest order in $\epsilon$,

$$\varphi_0=\varphi(0)-\tfrac{1}{4}s^2,\quad s=\epsilon\phi, \tag{31}$$

for $|s|<2\varphi(0)^{1/2}$ and periodically continued beyond with the period $4\varphi(0)^{1/2}$ in $s$. The plasma wave frequency is then $\Omega_p=\pi/2\varphi(0)^{-1/2}\epsilon$. It can also be expressed in terms of the average potential $\langle\varphi_0\rangle=2/3\varphi(0)$ or the average energy $\langle\gamma\rangle=1/2\langle\varphi_0\rangle$ using Eq. (7b). The approximation corresponds to the frequently used ultrarelativistic assumption $v_x=\pm 1$ or $p_x=\pm\gamma$. In this approximation the effect of the electromagnetic wave on the plasma wave is neglected. Because the electrostatic and the electromagnetic fields are calculated independently, an initial condition $\varphi'(0)\neq 0$ can be taken into account by merely shifting the parabola in $s$. The Fourier series of Eq. (31) is

$$\begin{aligned}\varphi_0=\varphi(0)-\frac{\pi^2}{12}+\cos(\Omega_p\phi)-\frac{\cos(2\Omega_p\phi)}{2^2}+\frac{\cos(3\Omega_p\phi)}{3^2}\\ -\dots\end{aligned} \tag{32}$$

so that the frequency spectrum contains a large set of higher harmonics.

The effect of the plasma wave on the electromagnetic wave can again be calculated by Eqs. (15b) and (16a),

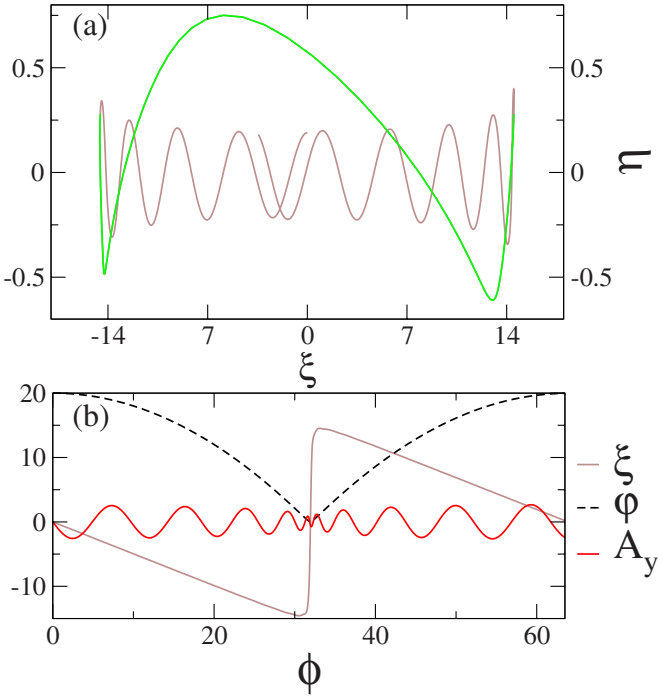


FIG. 4. (Color online) Coupling of an ultralarge amplitude electrostatic oscillation to an electromagnetic wave [$\epsilon=0.1$, $\varphi(0)=20$, and $A_y'(0)=-0.57$]. The results of the ultralarge amplitude approximation, e.g., $\hat{T}=63.1$ and $\xi_{\max}=15.8$, coincides well with the numerical results.

$$a_0(\phi) = g\left[\varphi(0) - \tfrac{1}{4}s^2\right]^{1/4}, \tag{33a}$$

$$\Omega(\phi) = \left[\varphi(0) - \tfrac{1}{4}s^2\right]^{-1/2}, \tag{33b}$$

$$f(\phi) = 2\epsilon^{-1}\arcsin(\Omega_{\mathrm{p}}\phi/\pi). \tag{33c}$$

Due to the Fourier series (32), the spectrum of $A_0$ contains multiple sidebands at $\Omega_{\mathrm{EM}} \pm n\Omega_{\mathrm{p}}$. The evaluation of Eq. (18) yields this time

$$\alpha_0 = \frac{4\varphi(0)}{\pi^2} = \frac{12}{\pi^2}\langle\gamma\rangle \tag{34}$$

so that the dispersion relation is

$$c^2k^2 = \omega^2 - \frac{\pi^2\omega_{\mathrm{p}}^2}{12\langle\gamma\rangle} \quad (*). \tag{35}$$

This time, the effective plasma frequency is reduced by a factor $\sqrt{12}/\pi$ in comparison to the periodic case. The term of order $\omega_{\mathrm{p}}^2$ in the dispersion relation deviates therefore about 20% from the corresponding term of the periodic or small amplitude case. Furthermore, now, the effective plasma frequency appearing in the dispersion relation equals the plasma wave frequency in the laboratory frame,

$$\hat{\Omega}_{\mathrm{p}} = \sqrt{\alpha}\Omega_{\mathrm{p}} = \frac{\pi}{2}\varphi(0)^{-1/2}\omega_{\mathrm{p}} = \frac{\pi}{\sqrt{12\langle\gamma\rangle}}\omega_{\mathrm{p}}. \tag{36}$$

Here, the modified plasma wave frequency results only from the nonlinearity of the plasma oscillation. Because in the parabolic approximation one has $\hat{\Omega}_{\mathrm{p}}=\pi/2/\xi_{\max}$, it follows for the longitudinal electron oscillation amplitude $\xi_{\max}=\varphi(0)^{1/2}/\omega_{\mathrm{p}}$.

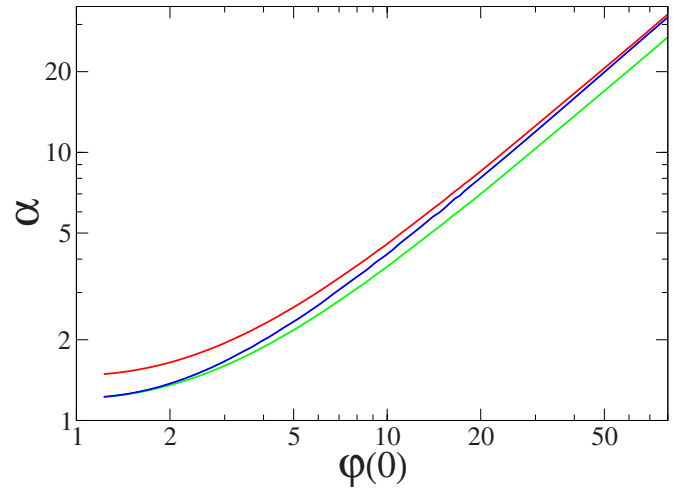


FIG. 5. (Color online) The development of the correction factor $\alpha$ of the dispersion relation, $c^2k^2=\omega^2-\omega_{\mathrm{p}}^2/\alpha$ $(*)$, as a function of the plasma wave amplitude for small densities: from the periodic case (with no plasma wave) over solutions with intermediate plasma waves up to solutions with an ultralarge plasma wave. The analytic solutions $\alpha=\langle\gamma\rangle$ and $\alpha=12/\pi^2\langle\gamma\rangle$ are the lower and upper boundaries [$\epsilon=0.01$, $a_0=1$, i.e., $A_y'(0)=-0.9$, $\varphi_{\mathrm{p}}=1.22$].

It should be noted that the parabolic approximation for an ultralarge amplitude electrostatic oscillation is applicable similarly for arbitrarily large plasma densities. However, the separation of the time scales, the coupling to the electromagnetic wave, and the dispersion relation are only investigated for small densities. In addition, at the critical density, a similar parabolic approximation can be used to describe a periodic ultralarge amplitude electromagnetic wave, which is almost transverse.[27]

In Fig. 4 the coupling of an ultralarge amplitude electrostatic oscillation to an electromagnetic wave is illustrated. Though the electromagnetic contribution is quite large, $A_{\max}=2.5$ for $\varphi_{\max}=20$, the ultralarge amplitude approximation for $\phi$ is applicable ($\hat{T}=63.1$, $\xi_{\max}=15.8$, and $\alpha=8.1$).

## VI. COUPLING TO AN INTERMEDIATE AMPLITUDE PLASMA WAVE

For a plasma wave of intermediate amplitude (in comparison to the electromagnetic amplitude), the analytical results from the previous sections are not applicable. Anyhow, $\alpha$ and the dispersion relation can still be numerically calculated from Eq. (18). A systematic investigation shows that for small densities, the value of $\alpha$ always increases monotonically from the periodic result $\langle\gamma\rangle$ to the ultralarge amplitude result $12/\pi^2\langle\gamma\rangle$ when the plasma wave amplitude is increased. The analytic results can therefore be used as a lower boundary of $\alpha$ (contribution mainly from the electromagnetic wave) and an upper boundary (contribution mainly from the ultra-large amplitude plasma wave). In Fig. 5 the exact dependence of $\alpha$ on the plasma wave amplitude is depicted for electromagnetic amplitude $a_0=1$. This allows to roughly estimate the deviation from $\alpha=\langle\gamma\rangle$, depending on the amplitude of the plasma wave.

## VII. CIRCULAR POLARIZATION

For circular polarization the periodic solutions are purely transverse and contain no higher harmonics. Then, the $\boldsymbol{v}\times\boldsymbol{B}$-force exactly cancels, and it is $\alpha=\gamma=\varphi=\sqrt{1+\frac{1}{2}a^2}$. Nevertheless, an initial longitudinal momentum leads to quasiperiodic solutions with phase and amplitude modulation. In this situation the electromagnetic wave again couples nonlinearly with the electrostatic wave through the modulated density and the relativistic gamma factor.

With the ansatz

$$A_y(s,f) = -\frac{a(s)}{\sqrt{2}}\sin(f), \tag{37a}$$

$$A_z(s,f) = \frac{a(s)}{\sqrt{2}}\cos(f), \tag{37b}$$

$$\varphi(s,f) = \varphi(s), \tag{37c}$$

and the separation of the two time scales, the basic equations (5a)–(5c) can be further simplified,

$$\epsilon^2 a(s)_{ss} + F(s)a(s) = \Omega(s)^2 a(s), \tag{38a}$$

$$2\Omega(s)a(s)_s + \Omega(s)_s a(s) = 0, \tag{38b}$$

$$\epsilon^2\varphi(s)_{ss} + F(s)\varphi(s) = \alpha, \tag{38c}$$

$$F(s) = \sqrt{\frac{1+\epsilon^2}{\varphi(s)^2 + \epsilon^2[1+a(s)^2/2]}}. \tag{38d}$$

Equations (38a) and (38b) follow from the differential equation of $A_y$ because the secular $\cos(f)$- and $\sin(f)$-components have both to be equal to zero. Equation (38b) is equivalent to

$$a(s)^2\Omega(s) = \beta = \text{constant}. \tag{39}$$

Therefore, for circular polarization $a^2\Omega$ is an exact invariant in all orders of perturbation theory.

The solution of Eqs. (38a) and (38c) is

$$\varphi_0(s)_{ss} + \tfrac{1}{2}\left(1 - \varphi_0^{-2} - \tfrac{1}{2}g_0^2\varphi_0^{-3/2}\right) = 0, \tag{40a}$$

$$\varphi_1(s,f) = 0, \tag{40b}$$

$$\begin{aligned}\varphi_2(s)_{ss} &+ \tfrac{\varphi_2}{\varphi_0^3}\left(1 + \tfrac{3}{8}g_0^2\varphi_0^{1/2}\right) - \tfrac{1}{2}g_0 g_2\varphi_0^{-3/2} \\ &- \tfrac{1}{8}\left(1 + g_0^2\varphi_0^{-3/2} + 2\varphi_0^{-2} + \tfrac{3}{16}g_0^2 E_0\varphi_0^{-5/2}\right. \\ &\left. - C_1 g_0^4\varphi_0^{-3} - \tfrac{45}{16}g_0^2\varphi_0^{-7/2} - 3\varphi_0^{-4}\right) = 0,\end{aligned} \tag{40c}$$

$$\varphi_3(s) = 0, \quad \varphi_4(s) \neq 0, \tag{40d}$$

$$a_0(s) = g_0\varphi_0^{1/4}, \tag{41a}$$

$$a_1(s) = 0, \tag{41b}$$

$$\begin{aligned}a_2(s) &= \tfrac{1}{64}g_0(3E_0\varphi_0^{-3/4} + C_2 g_0^2\varphi_0^{-5/4} + 3\varphi_0^{-7/4}) \\ &+ \tfrac{1}{4}g_0\varphi_2\varphi_0^{-3/4} + g_2\varphi_0^{1/4},\end{aligned} \tag{41c}$$

$$a_3(s) = 0, \quad a_4(s) \neq 0, \tag{41d}$$

$$\Omega_0(s) = \sqrt{\frac{1}{\varphi_0}}, \tag{42a}$$

$$\Omega_1(s) = 0, \tag{42b}$$

$$2\frac{\Omega_2(s)}{\Omega_0} = -\frac{\varphi_2}{\varphi_0} + \frac{9}{16} - \frac{3}{16}\varphi_0^{-2} - \frac{3}{16}E_0\varphi_0^{-1} - C_3 g_0^2\varphi_0^{-3/2}, \tag{42c}$$

$$\Omega_3(s) = 0. \tag{42d}$$

All perturbative corrections are a result of amplitude and phase modulation. In comparison with the case of linear polarization, the higher harmonics are missing. All the terms of the solution for circular polarization (40a)–(42d) are already present in the solution for linear polarization (14a)–(16d) and differ only in the coefficients $C_1=3/4$, $C_2=0$, and $C_3=0$. Again, the integration constants $g_0$ and $g_2$ are fixed by the initial condition $a_0(0)=a(0)$.

Due to the same structure of the lowest order expressions, the results for the dispersion relation from the last sections apply also for circular polarization. Especially, the dispersion relation

$$c^2k^2 = \omega^2 - \frac{\omega_p^2}{\alpha} \quad (*)$$

differs also for circular polarization from the periodic result $\alpha=\gamma$ if a plasma wave is coupled to the electromagnetic wave. Again, the analytic solutions $\alpha=\langle\gamma\rangle$ and $\alpha=12/\pi^2\langle\gamma\rangle$ are in the order $\omega_p^2$ lower and upper boundaries for $\alpha$.

## VIII. COMPARISON WITH PIC SIMULATIONS

All the results from above are derived in the framework of the Akhiezer–Polovin model. However, these results can also be independently compared with PIC simulations, which in addition include particle effects. For the comparison a relativistic one-dimensional (1D)-PIC simulation is used with several million simulation particles so that a high accuracy is achieved. In the simulation various solutions of the Akhiezer–Polovin model are used as initial conditions and are afterward propagated in the plasma. In a systematic comparison for small densities, all solutions are found to propagate stably at least over some hundred vacuum wavelengths. Therefore, the solutions of the Akhiezer–Polovin model are also stationary waves in PIC simulations, and the analytical and numerical solutions of this work are confirmed with great accuracy. A typical simulation is depicted in Fig. 6 for linear polarization. There is no appreciable change in the shape of the solution after the propagation. Naturally, deviations from the 1D behavior are expected in two- and three-dimensional simulations. However, these effects are beyond the scope of the present work.

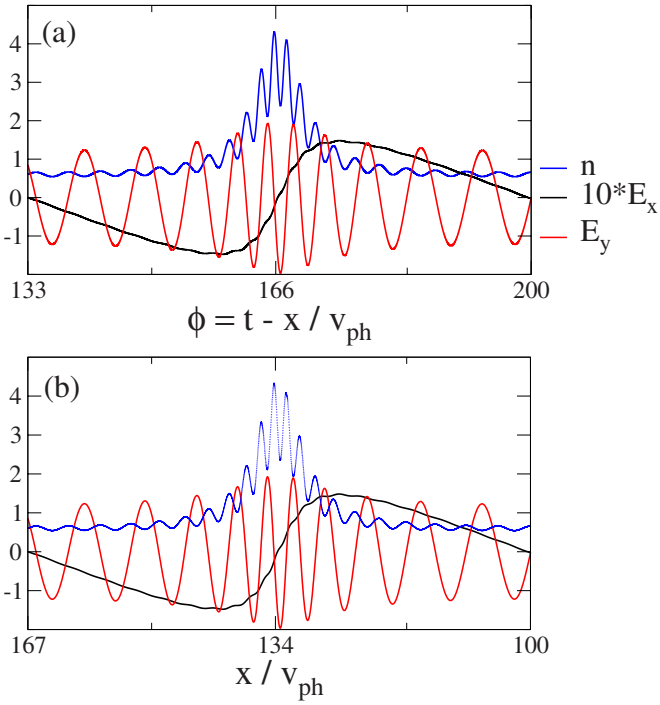


FIG. 6. (Color online) Comparison of a quasiperiodic solution of the Akhiezer–Polovin model (a) with a 1D-PIC simulation with $2\times10^6$ particles (b). In the PIC simulation the quasiperiodic solution of (a) has been propagated from $t=0$ to $t=300$, i.e., roughly over 50 vacuum wavelengths, without changing shape. [linear polarization, $\epsilon=0.1$, $a_0=1$, i.e., $A_y'(0)=-0.9$, $\varphi(0)=1.35$, $\alpha=1.82$].

## IX. SUMMARY AND DISCUSSION

In this work we have studied the propagation of coupled electromagnetic and electrostatic waves in relativistic plasmas for small densities. A two-time-scale method has been used to solve the Akhiezer–Polovin model analytically up to the fourth order in the plasma density. A general method for the derivation of the dispersion relation has been introduced. The method has been applied to study the influence of the coupling on the dispersion relation and to answer the long-standing question of the correct relativistic modification of the plasma frequency. In two important limiting cases, analytical dispersion relations have been calculated considering the full plasma response. The investigation of the general case shows that the analytical results for the corrections of the plasma frequency can be used as lower and upper boundaries of the general case. Finally, the results for linear polarization have been shown to apply also largely for circular polarization. Especially, the dispersion relation differs from the common one if a plasma wave is present.

All the results in this work hold only for stationary solutions. For example, in wake field generation an electrostatic wake field is excited with the group velocity $v_g$ of the electromagnetic wave, $v_{ph}^{ES}=v_g^{EM}\approx 1/v_{ph}^{EM}$. In this situation the frequency of a small amplitude wake field in a long electromagnetic wave train is found in PIC simulations to be $\hat{\Omega}_p\approx\varphi_p^{-1/2}\omega_p$. The frequency differs from $\omega_p$, which one finds if the wake field is excited behind a short pulse, but it also differs from Eq. (28). This is due to the nonstationarity of the situation and the resulting difference in the coupling to the electromagnetic wave train. However, for stationary solutions we find in PIC simulations exactly the behavior predicted in this work.

## ACKNOWLEDGMENTS

This work was supported in part by the German Research Foundation (DFG) and by a doctoral research grant D/08/44316 of the German Academic Exchange Service (DAAD).

## APPENDIX: DERIVATION OF THE BASIC SET OF EQUATIONS

The basic set of equations for the potentials is derived from the Maxwell equations and the relativistic fluid equations. Similar derivations are found in Refs. 22 and 34. The Maxwell equations for the potentials in Coulomb gauge, the continuity equation, and the relativistic equation of motion are in dimensionless form

$$(\partial_t^2-\Delta)\boldsymbol{A}=\boldsymbol{j}=-\omega_p^2 n\,\boldsymbol{v}, \tag{A1a}$$

$$\Delta\varphi=-\tau=\omega_p^2(n-1), \tag{A1b}$$

$$\partial_t n+\nabla\cdot(n\boldsymbol{v})=0, \tag{A1c}$$

$$\frac{d}{dt}\boldsymbol{p}=-(-\nabla\varphi-\partial_t\boldsymbol{A}+\boldsymbol{v}\times\nabla\times\boldsymbol{A}), \tag{A1d}$$

where $\boldsymbol{p}=\gamma\boldsymbol{v}$ and $\gamma=1/\sqrt{1-v^2}=\sqrt{1+p^2}$. In a 1D geometry $[\boldsymbol{A}(x,t)=\boldsymbol{A}(x,t),\ldots]$ in Coulomb gauge, one can choose $\boldsymbol{A}_x=0$. Then, the equation of motion [Eq. (A1d)] for the transverse direction leads to

$$\boldsymbol{p}_\perp=\boldsymbol{A}_\perp. \tag{A2}$$

In the longitudinal direction it is

$$\frac{d}{dt}p_x=\partial_x\varphi-\boldsymbol{v}_\perp\cdot\partial_x\boldsymbol{A}_\perp, \tag{A3}$$

which reduces with $\partial_x\gamma=\boldsymbol{v}\cdot\partial_x\boldsymbol{p}$ to

$$\partial_t p_x=\partial_x(\varphi-\gamma). \tag{A4}$$

Assuming the stationary situation of the Akhiezer–Polovin problem, i.e., all quantities depend only on the phase $\hat{\phi}=t-x/v_{ph}$, the partial derivatives can be replaced by

$$\partial_t\to\frac{d}{d\hat{\phi}},\quad \partial_x\to-\frac{1}{v_{ph}}\frac{d}{d\hat{\phi}}. \tag{A5}$$

Then, Eqs. (A1c) and (A4) can be integrated, yielding

$$n(v_{ph}-v_x)=v_{ph}, \tag{A6}$$

$$\gamma-v_{ph}p_x=\varphi, \tag{A7}$$

where $n=1$ and $\varphi=1$ are assumed for $\boldsymbol{p}=0$. Equation (A7) together with $\gamma=\sqrt{1+p_x^2+A^2}$ can be solved for $p_x$,

$$p_x = \frac{1}{v_{\rm ph}^2 - 1}\left(\sqrt{\varphi^2 + (v_{\rm ph}^2 - 1)(1 + A^2)} - v_{\rm ph}\varphi\right)$$

$$= \frac{v_{\rm ph}}{v_{\rm ph}^2 - 1}(1/F - \varphi), \tag{A8a}$$

$$F = \frac{v_{\rm ph}}{\sqrt{\varphi^2 + (v_{\rm ph}^2 - 1)(1 + A^2)}}. \tag{A8b}$$

Similar all other quantities can be expressed by the potentials, e.g.,

$$\gamma = \varphi + v_{\rm ph} p_x = \frac{1}{v_{\rm ph}^2 - 1}(v_{\rm ph}^2/F - \varphi), \tag{A9a}$$

$$E_x = \frac{1}{v_{\rm ph}}\frac{d}{d\hat{\phi}}\varphi, \quad E_y = -\frac{d}{d\hat{\phi}}A_y. \tag{A9b}$$

Especially, Eq. (A6) yields

$$n - 1 = \frac{1}{v_{\rm ph}^2 - 1}(1 - F\varphi), \tag{A10a}$$

$$\frac{n}{\gamma} = F. \tag{A10b}$$

Using Eqs. (A10a) and (A10b), the Maxwell equations (A1a) and (A1b),

$$\left(1 - \frac{1}{v_{\rm ph}^2}\right)\frac{d^2}{d\hat{\phi}^2}\boldsymbol{A} = -\omega_{\rm p}^2\frac{n}{\gamma}\boldsymbol{A}, \tag{A11a}$$

$$\frac{1}{v_{\rm ph}^2}\frac{d^2}{d\hat{\phi}^2}\varphi = \omega_{\rm p}^2(n - 1), \tag{A11b}$$

reduce to

$$\frac{d^2}{d\hat{\phi}^2}\boldsymbol{A} + \alpha F\boldsymbol{A} = 0, \tag{A12a}$$

$$\frac{d^2}{d\hat{\phi}^2}\varphi + \alpha(F\varphi - 1) = 0, \tag{A12b}$$

where $\alpha = \omega_{\rm p}^2 v_{\rm ph}^2/(v_{\rm ph}^2 - 1)$. With $\epsilon = \sqrt{v_{\rm ph}^2 - 1}$ and with the derivative with respect to the new phase $\phi = \sqrt{\alpha}\hat{\phi}$, Eqs. (A12a) and (A12b) correspond to the basic equations for the potentials [Eqs. (5a) and (5b)].


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